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New Type of Exactly Solvable Potential

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It is shown that, starting from an exactly solvable potential and making use of the theory of a system of coupled differential equations, it is possible to construct a new type of second-generation potential which is also exactly solvable.

1. INTRODUCTION

A potential $V(x)$ is exactly solvable when the set of eigenfunctions $\{\psi_n\}$ and eigenvalues $\{E_n\}$ of the equation

$$H\psi_n = E_n\psi_n, \qquad H = \frac{d^2}{dx^2} - V(x) \tag{1}$$

can be reached entirely by algebraic means.

It will be shown in this paper that each exactly solvable potential in turn may generate another set of solvable potentials $V_n(x)$, where the number n refers to the $|n\rangle$ eigenstate of the original potential $V(x)$. The present work uses some recent results obtained from the investigation of a system of coupled differential equations of first order (Cao, 1994), which will be adapted to the conditions of the subject under consideration.

Therefore, for clarity, it seems useful to begin this presentation with a brief review of some features of this theory which will be extensively implemented in the discussion below.

As a first application, a simple discussion shows that the conventional supersymmetrization $SU(2)$ turns out to be one of the consequences of the present approach.

A second application will be discussed in the Appendix with a simple example which underlines how the above results can be handled in practice.

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2. THE SYSTEM OF COUPLED DIFFERENTIAL EQUATIONS

Consider the matrix equation

$$\phi' + F\phi = 0$$
$$\phi = (\phi_1, \phi_2)^+, \qquad \phi' = \frac{d\phi}{dx}, \qquad F = \begin{pmatrix} u_1 & d_1 \\ d_2 & u_2 \end{pmatrix} \tag{2}$$

u_i, d_i may be any analytic functions. The theory consists in defining a mixing function X, $\phi_1 = -X\phi_2$, which enables one to write explicitly the system (2) as

$$\phi_2' + \left(\frac{X'}{X} + u_1 - \frac{d_1}{X}\right)\phi_2 = 0; \qquad \phi_2' + (u_2 - d_2X)\phi_2 = 0 \tag{3}$$

Compatibility of these two equations requires that the quantity X must be a solution of the following first-order differential equation:

$$d_1 = X' - (u_2 - u_1)X - d_2X^2 \tag{4}$$

Differentiation of the system, on the other hand, leads to a system of two coupled differential equations of second order:

$$\begin{aligned} \phi_1'' - (u_1^2 - u_1^1 + d_1d_2)\phi_1 + (d_1^1 - d_1(u_1 + u_2))\phi_2 &= 0 \\ \phi_2'' - (u_2^2 - u_2^1 + d_1d_2)\phi_2 + (d_2^1 - d_2(u_1 + u_2))\phi_1 &= 0 \end{aligned} \tag{5}$$

The set of three equations (4), (5) is sufficient for the needs of the ensuing development. Note, however, that the system of coupled equations of type (5) had already been investigated previously from a different point of view where a theorem on the complete separation of the equations (decoupling) was formulated setting up some specific constraints (Cao, 1981). In the present work, we shall nevertheless exclude these possibilities and consider instead another construction in which one of the coupling terms (d_1 or d_2) is equal to zero. As the case $u_1 \neq u_2$ does not in fact bring anything new in the discussion below, we shall therefore assume that

$$u_1 = u_2 = u; \qquad d_2 = 0 \tag{6}$$

Theorem I. If the potential $V(x) = u^2 - u^1$ is exactly solvable, then to any couple of its eigenfunction $|m\rangle$, $|n\rangle$ $(m \geq n)$ it is always possible to associate a system of coupled equations of type (2) with constraint (6).

Proof. It will be more convenient to proceed in two steps.

(i) Consider first the couple (m, $n = 0$); there is no loss of generality by assuming $E_0 = 0$. Using now relation (4) and the definition of the mixing function, write the first equation of (5) as

$$\phi''_{1,m} - (u^2 - u^1)\phi_{1,m} = \frac{1}{X_m}(X''_m - 2uX'_m)\phi_{1,m}$$

Let E_m be the eigenvalue corresponding to the state $|m\rangle$, and assume that X_m is solution of the following second-order differential equations:

$$X''_m - 2uX'_m - E_m X_m = 0 \tag{7}$$

Then

$$\phi''_{1,m} - V\phi_{1,m} = E_m\phi_{1,m}; \qquad \phi''_2 - V\phi_2 = 0$$

which clearly indicates that the second component corresponds to the zero state, while the first one represents the $|m\rangle$ state of the same potential $V(x)$.

(ii) Generalizing to the couple (m, n) ($m > n$), the same reasoning remains valid, but with a slight modification. Write system (1) in the form

$$\phi' + F_n\phi = 0 \tag{8}$$

where F_n is defined as

$$F_n = \begin{pmatrix} u - \dfrac{X'_n}{X_n} & d_{m,n} \\ 0 & u - \dfrac{X'_n}{X_n} \end{pmatrix}$$

X_n is the solution of equation (7) as discussed above and the new coupling term is $d_{m,n}$, with obvious meaning of the indices.

Introduce now another mixing function $\lambda_{m,n}$ such that $\phi_{1,m} = -\lambda_{m,n}\phi_{2,n}$. A compatibility condition similar to (4) applied to this case leads to $d_{m,n} = \lambda'_{m,n}$. Therefore, if the mixing function $\lambda_{m,n}$ is a solution of the equation

$$\lambda''_{m,n} - 2\left(u - \frac{X'_n}{X_n}\right)\lambda'_{m,n} - A_{m,n}\lambda_{m,n} = 0 \tag{9}$$

$A_{m,n}$ is a constant parameter; hence for the two components $\phi_{1,m,n}$ and $\phi_{2,n}$ we may write

$$\phi''_{1,m,n} - V_n\phi_{1,m,n} = A_{m,n}\phi_{1,m,n}; \qquad \phi''_{2,n} - V_n\phi_{2,n} = 0$$

in which

$$V_n = v_n^2 - v_n^1; \qquad v_n = \left(u - \frac{X_n^1}{X_n}\right)$$

Equation (9) is solvable with solution

$$\lambda_{m,n} = \frac{X_m}{X_n} \tag{10}$$

and $A_{m,n} = E_m - E_n$. The simplest way to see this is to proceed by substituting (10) in (9) and, after simplification, noting that

$$\lambda''_{m,n} - 2\left(u - \frac{X'_n}{X_n}\right)\lambda'_{m,n} = \frac{1}{X_n}(X''_m - 2uX'_m) - \frac{X_m}{X_n^2}(X''_n - 2uX'_n)$$

Using now relation (7) hence confirms the above assertion.

The analytical form of the eigenfunctions is

$$\phi_{1,m,n} = -\lambda_{m,n}\phi_{2,n} \qquad \left(\phi_{2,n} = X_n \exp - \int u\, dx\right) \tag{11}$$

with eigenvalue spectrum $\{E_m - E_n\}$. This completes the proof of the theorem, which may lead to useful applications in physics. We shall now examine one of them by revisiting the conventional concept of supersymmetrization, $SU(2)$.

3. SUPERSYMMETRIZATION

Consider now two systems of coupled equations of the preceding type, i.e., type (2):

$$\bar{\phi}' + \bar{F}\bar{\phi} = 0; \qquad \phi' + F\phi = 0$$

$$\bar{\phi}' = (\bar{\phi}_1, \phi_0)^+; \qquad \bar{F} = \begin{pmatrix} -u & \bar{d} \\ 0 & u \end{pmatrix}; \qquad \phi' = (\phi_2, \phi_0)^+; \qquad F = \begin{pmatrix} u & d \\ 0 & 0 \end{pmatrix} \tag{12}$$

u, $\bar{d}$, d may be any analytic functions. First note that the second system in (12) is exactly the same as the case discussed above in part (i) of the proof, which means that all the results obtained there can be transposed here without difficulty.

Thus the new situation comes from the first system, which requires the definition of a new mixing function Y

$$\bar{\phi}_1 = -Y\phi_0$$

From the condition of compatibility (4), we have here

$$\bar{d} = Y' - 2uY \tag{13}$$

Hence, if the mixing function Y is solution of the equation

$$Y''_m - 2uY'_m - (2u' + B_m)Y_m = 0 \tag{14}$$

B_m being a constant parameter, then the final form of the first equation in (5) can be written as

$$\overline{\phi}''_{1,m} - \overline{V}\overline{\phi}_{1,m} = B_m\overline{\phi}_{1,m} \tag{15}$$

with $\overline{V} = u^2 + u^1$.

Returning now to the equation (7), by differentiation we have

$$X'''_n - 2uX''_n - (2u' + E_n)X'_n = 0 \tag{16}$$

The two equations (14) and (16) are identical if

$$\text{(a)} \quad Y_m = \alpha X'_n, \qquad \alpha \quad \text{constant}; \qquad \text{(b)} \quad B_m = E_n \tag{17}$$

On the other hand, for the zero state, $n = 0$, $X_0 = I$, so that the eigenfunction $\overline{\phi}_{1,0}$ is not defined. The only possibility is to set $m = n + 1$, $B_{n+1} = E_n$, or, in other words, the quantity B_m can be regarded as the eigenvalue corresponding to the state $\overline{\phi}_{1,m}$ for the potential $\overline{V}$ with

$$\overline{\phi}_{1,m} \simeq Y_m\phi_0 \qquad \left(\phi_0 \simeq \exp - \int u \, dx\right)$$

The present method therefore leads to the construction of another exactly solvable potential $\overline{V}$ from the former one V.

This situation presents a striking similarity with supersymmetrization [in the sense of Witten (1981)], which itself is based on the graded Lie algebra which accounts for the nilpotency of two operators Q, Q^+ (i.e., $Q^2 = Q^{+2} = 0$)

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}; \qquad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}; \qquad A\pm = \frac{d}{dx} \pm u(x)$$

with the Hamiltonian

$$H = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix} = \{Q, Q^+\}; \qquad H\pm = \frac{d^2}{dx} - (u^2 \pm u')$$

The two components $\overline{\phi}_1$, ϕ_2 are connected by the operators $A^\pm$

$$A^-\phi_2 = E^{1/2}\overline{\phi}_1, \qquad A^+\overline{\phi}_1 = -E^{1/2}\phi_2 \qquad (E\text{: eigenvalue})$$

The equivalence between the two approaches may be appreciated with the following remark:

$$\overline{\phi}_1 = -Y\phi_0 = \frac{1}{E^{1/2}} A^- X\phi_0 = \frac{1}{E^{1/2}} \left[\frac{X'}{X} + \frac{\phi'_0}{\phi_0} - u\right] X\phi_0$$

since $\phi_0'/\phi_0 = u$; hence

$$-Y\phi_0 = \frac{1}{E^{1/2}} X'\phi_0 \qquad \text{or} \qquad Y = \alpha X' \qquad (\alpha = -1/E^{1/2})$$

as predicted in (17).

This result in fact can be reached as revealing an equivalence between the present approach and conventional supersymmetrization and may provide a useful tool to get a deeper insight from a more general point of view, as will be seen in the discussion below.

Theorem II. If the potential $V = u^2 - u^1$ is exactly solvable, then it is always possible, corresponding to any eigenstate $|n\rangle$ of V, to construct a set of exactly solvable potentials of second generation $\bar{V}_n$.

Proof. The meaning of the term $\bar{V}_n$ will be clarified below; we begin by dealing with the mathematical aspect of the theorem. Consider two other systems of coupled equations:

$$\bar{\phi}' + \bar{F}_n\bar{\phi} = 0, \qquad \phi' + F_n\phi = 0$$

$$\bar{\phi} = (\bar{\phi}_1, \phi_0)^+; \qquad \bar{F}_n = \begin{pmatrix} -\left(u - \dfrac{X_n'}{X_n}\right) & \bar{d}_{m,n} \\ 0 & \left(u - \dfrac{X_n'}{X_n}\right) \end{pmatrix}; \tag{18}$$

$$\phi = (\phi_2, \phi_0); \qquad F_n = \begin{pmatrix} u - \dfrac{X_n'}{X_n} & d_{m,n} \\ 0 & u - \dfrac{X_n'}{X_n} \end{pmatrix}$$

X_n, $d_{m,n}$ are already defined above; the new coupling terms $\bar{d}_{m,n}$, may be any analytic functions depending on the couple of indices (m, n).

Note first that the second system in (18) is identical to the one discussed in part (ii) of the proof of Theorem I, so that we have $\phi_2 = -\lambda_{m,n}\phi_0$ and the equation (9) for $\lambda_{m,n}$ remains valid here. To deal with the first system, introduce a new mixing function $\bar{\lambda}_{m,n}$ defined as $\bar{\phi}_{1,m,n} = -\bar{\lambda}_{m,n}\phi_0$ and follow exactly the same reasoning. Compatibility conditions similar to (4) and (13) lead to the condition

$$\bar{d}_{m,n} = \bar{\lambda}_{m,n}' - 2\left(u - \frac{X_n'}{X_n}\right)\bar{\lambda}_{m,n} \tag{19}$$

Therefore, $C_{m,n}$ being a constant parameter, if $\bar{\lambda}$ is a solution of the second-order equation

$$\bar{\lambda}''_{m,n} - 2\left(u - \frac{X'_n}{X_n}\right)\bar{\lambda}'_{m,n} - \left[2\left(u' - \left(\frac{X'_n}{X_n}\right)'\right) + C_{m,n}\right]\bar{\lambda}_{m,n} = 0 \quad (20)$$

then the Schrödinger equation corresponding to the first system of (18) can be written as ($v_n = u - X'_n/X_n$)

$$\bar{\phi}_{1,m,n} - (v_n^2 + v'_n)\bar{\phi}_{1,m,n} = C_{m,n}\bar{\phi}_{1,m,n}$$

To solve (20), return again to (9), so that after differentiation

$$\lambda'''_{m,n} - 2\left(u - \frac{X'_n}{X_n}\right)\lambda''_{m,n} - \left[2\left(u' - \left(\frac{X'_n}{X_n}\right)'\right) + A_{m,n}\right]\lambda'_{m,n} = 0 \quad (21)$$

which is identical to (20) and provides a relationship between $\bar{\lambda}$ and λ if

$$\text{(a)} \quad \bar{\lambda}_{m,n} = \beta\lambda'_{m,n}, \quad \beta = \text{const}; \qquad \text{(b)} \quad C_{m,n} = A_{m,n}$$

Note that for $m = n$, $\lambda_{n,n} = 1$, so that $\bar{\phi}_{1,n,n}$ is not defined and we must set $m = n + 1$. The eigenfunctions $\bar{\phi}_{1,m,n}$, $\phi_{2,m,n}$ are

$$\bar{\phi}_{1,m,n} \simeq \bar{\lambda}_{m,n}\phi_0, \qquad \phi_{2,m,n} \simeq \lambda_{m,n}\phi_0 \quad (22)$$

Making explicit the analytical form of $\bar{V}_n$ and making use of (7), we can write the final form of $\bar{V}_n$ as

$$\bar{V}_n = u^2 + u' + 2\frac{X'_n}{X_n}\left[\frac{X'_n}{X_n} - 2u\right] \quad (23)$$
$$\bar{\phi}''_{1,m,n} - \bar{V}_n\bar{\phi}_{1,m,n} = (E_m - 2E_n)\bar{\phi}_{1,m,n}$$

We interpret this as follows:

- $\bar{V}_n$ is an exactly solvable potential with eigenspectrum $\{E_m - 2E_n\}$ and eigenfunctions defined by (22).
- For $n = 0$ the potential $\bar{V}_0$ becomes identical to $\bar{V} = u^2 + u^1$, revealing that, in the context of the present theory, supersymmetrization can be regarded as a special case corresponding to $n = 0$.
- For each e.s. potential $V(x)$ and corresponding to every eigenstate $|n\rangle$, it is possible to construct a second generation e.s. potential $\bar{V}_n$; the meaning of this term refers to the presence of the index n.
- The domain of definition of $\bar{V}_n$ may be more restricted and depends on the analytical structure of X_n, since n represents the number of nodes in $\phi_{2,n}$ corresponding to possible singularities for $\bar{V}_n$.

Conversely, it can be shown that the function $\overline{\phi}_{1,m,n}$ given by (22) is effectively an eigenfunction of the potential $\overline{V}_n$ with eigenvalue $E_m - 2E_n$.

In fact, we may write

$$\frac{\overline{\phi}''_{1,m,n}}{\overline{\phi}_{1,m,n}} = \frac{\phi''_0}{\phi_0} - 2v_n \frac{\lambda''_{m,n}}{\lambda'_{m,n}} + \frac{\lambda'''_{m,n}}{\lambda'_{m,n}}$$

where $\phi''_0/\phi_0 = u^2 - u^1 + E_n$ and successive use of (14) and (16) effectively leads to (23).

4. CONCLUSION

The theory of a system of two coupled equations of type (2) with constraint (6) enables one to generate three types of exactly solvable potentials:

- The "parent" potential $V = u^2 - u'$.
- Its partner $\overline{V} = u^2 + u'$. Their respective eigenspectra show double degeneracy, except for the case $n = 0$.
- The second-order generation potential, for which the eigenspectrum is related to the original one $\{E_m - 2E_n\}$.

A simple example will be given in the Appendix in order to show how the method can be implemented in practice.

There is in fact a fourth type of potential which is not discussed in this work, since they are not exactly, but only quasi-exactly solvable, and require a different analytical treatment. This question will be presented in another paper.

Finally, it may also be interesting to speculate on whether this process of filiation could be extended to higher order generations. This is discussed in Appendix B.

APPENDIX A

Let $u = A$ cotanh x, where A is an arbitrary constant. For our purpose it will be sufficient to consider the special case $A = -1/2$, for which equation (7) reduces to the usual Legendre equation,

$$X''_n + \text{cotanh}\, x\, X'_n + E_n X_n = 0$$

with solution $X_n = P_n(\text{cosinh}\, x)$, where P_n are Legendre polynomials, and $E_n = -n(n + 1)$, $n = 0, 1, 2, \ldots$. The potentials V and $\overline{V}$ are

$$V = \frac{1}{4}\left(1 - \frac{1}{\sinh^2 x}\right); \qquad \overline{V} = \frac{1}{4}\left(1 + \frac{3}{\sinh^2 x}\right)$$

while the second-generation potential $\overline{V}_n$ is given in (23). Their analytical form and eigenspectrum are summarized in Table I for the first values of n.

Table I. Second-Generation Potential $\overline{V}_n$ and Eigenspectrum[a]

n	$\overline{V}_n$	$C_{m,n}$
1	$\frac{1}{4}\left(1 + \frac{3}{\sinh^2 x}\right) - \frac{2}{\cosh^2 x} + 4$	$m(m+1) - 4,\ m = 2, 3, \ldots$
2	$\frac{1}{4}\left(1 + \frac{z}{\sinh^2 x}\right) + 12P(z)$	$m(m+1) - 12,\ m = 3, 4, \ldots$
	$P(z) = \frac{z^2}{(3z^2-1)^2}(9z^2 - 7)$	
3	$\frac{1}{4}\left(1 + \frac{z}{\sinh^2 x}\right) + 2P(z)$	$m(m+1) - 24,\ m = 4, 5, \ldots$
	$P(z) = \frac{1}{z^2(5z^2-1)^2}(300z^6 - 375z^4 + 108z^2 - 9)$	

[a] $z = \cosh x$.

Note that for the case $A \neq -1/2$, it can be shown that the solution of equation (7) can be expressed in terms of a hypergeometric series which involves A, so that V_n depends now on two parameters n and A, i.e., $V(x, A)_n$. For instance, with $n = 1$, we find

$$\overline{V}(x, A)_1 = A^2 + 2(1 - 2A) + \frac{A(A+1)}{\sinh^2 x} - \frac{2}{\cosh^2 x}$$

$$C_{m,1} = m^2 - 2A(m-2) - (A^2 + 2)$$

It can be verified that for the special case $A = -1/2$, this agrees with the result given in Table I up to a constant.

APPENDIX B

A third-order generation potential involves two indices n, m which correspond to the $|n\rangle$, $|m\rangle$ eigenstates of the parent potential V and will be denoted by $\overline{V}_{m,n}$.

Consider again the system (18) in which the couple $\overline{F}_n$, F_n are replaced by the couple $\overline{F}_{m,n}$ and $F_{m,n}$, and $\overline{d}_{m,n}$, $d_{m,n}$ by $\overline{d}_{s,m,n}$, $d_{s,m,n}$:

$$\overline{F}_{m,n} = \begin{pmatrix} -\left[u - \left(\frac{X_n'}{X_n} + \frac{X_m'}{X_m}\right) & \overline{d}_{s,m,n} \\ 0 & u - \left(\frac{X_n'}{X_n} + \frac{X_m'}{X_m}\right) \end{pmatrix}; \qquad F_{m,n}$$

$$= \begin{pmatrix} u - \left(\frac{X_n'}{X_m} + \frac{X_m'}{X_m}\right) & d_{s,m,n} \\ 0 & u - \left(\frac{X_n'}{X_n} + \frac{X_m'}{X_m}\right) \end{pmatrix}$$

Introduce the mixing functions $\bar{\lambda}_{s,m,n}$ and $\lambda_{s,m,n}$ such that

$$\bar{\phi}_{1,s,m,n} = -\bar{\lambda}_{s,m,n}\phi_0, \qquad \phi_{2,s,m,n} = -\lambda_{s,m,n}\phi_0$$

Similar to (9), the second-order differential equation for $\lambda_{s,m,n}$ is

$$\lambda''_{s,m,n} - 2\left[u - \left(\frac{X'_n}{X_n} + \frac{X'_m}{X_m}\right)\right]\lambda'_{s,m,n} - B_{s,m,n}\lambda_{s,m,n} = 0$$

$B_{s,m,n}$ being a constant parameter.

It can then be shown that its solution is

$$\lambda_{s,m,n} = \frac{X_s}{X_m X_n}$$

The algebra here become fairly cumbersome and will not be displayed in this Appendix. We quote instead the final result, which is, however, simple:

$$\bar{V}_{m,n} = u^2 + u^1 + 2 \sum_{i=m,n} \frac{X'_i}{X_i}\left(\frac{X'_i}{X_i} - 2u\right)$$

$$B_{s,m,n} = E_s - 2 \sum_{i=m,n} E_i$$

indicating that extension to potentials of higher order generations is also possible.

REFERENCES

Cao, X. C. (1981). *Journal of Physics A: Mathematical and General,* **14**, 1069.

Cao, X. C. (1994). *Comptes Rendus de l'Academie des Sciences Paris,* **319**(11), 625.

Witten, E. (1981). *Nuclear Physics B,* **185**, 131.